THE ERDŐS PARADOX

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Prologue

The great Hungarian mathematician Paul Erdős was born in Budapest on March 26, 1913. He died alone in a hospital room in Warsaw, Poland, on Friday afternoon, September 20, 1996. It was sad and ironic that he was alone, because he probably had more friends in more places than any mathematician in the world. He was in Warsaw for a conference. Vera Sós had also been there, but had gone to Budapest on Thursday and intended to return on Saturday with András Sárközy to travel with Paul to a number theory meeting in Vilnius. On Thursday night Erdős felt ill and called the desk in his hotel. He was having a heart attack and was taken to a hospital, where he died about 12 hours later. No one knew he was in the hospital. When Paul did not appear at the meeting on Friday morning, one of the Polish mathematicians called the hotel. He did not get through, and no one tried to telephone the hotel again for several hours. By the time it was learned that Paul was in the hospital, he was dead.

Vera was informed by telephone on Friday afternoon that Paul had died. She returned to Warsaw on Saturday. It was decided that Paul should be cremated. This was contrary to Jewish law, but Paul was not an observant Jew and it is not known what he would have wanted. Nor was he buried promptly in accordance with Jewish tradition. Instead, four weeks later, on October 18, there was a secular funeral service in Budapest, and his ashes were buried in the Jewish cemetery in Budapest.

Erdős strongly identified with Hungary and with Judaism. He was not religious, but he visited Israel often, and established a mathematics prize and a post-doctoral fellowship there. He also established a prize and a lectureship in Hungary. He told me that he was happy whenever someone proved a beautiful theorem, but that he was especially happy if the person who proved the theorem was Hungarian or Jewish.

Mathematicians from the United States, Israel, and many European countries travelled to Hungary to attend Erdos's funeral. The following day a conference, entitled "Paul Erdős and his Mathematics," took place at the Hungarian Academy of Sciences in Budapest, and mathematicians who were present for the funeral were asked to lecture on different parts of Erdős's work. I was asked to chair one of the sessions, and to begin with some personal remarks about my relationship with Erdős and his life and style.

This paper is in two parts. The first is the verbatim text of my remarks at the Erdős memorial conference in Budapest on October 19, 1996. A few months after the funeral and conference I returned to Europe to lecture in Germany. At Bielefeld someone told me that my eulogy had generated controversy, and indeed, I heard the same report a few weeks later when I was back in the United States. Eighteen years later, on the 100th anniversary of his birth, it is fitting to reconsider Erdős's life and work.

1. Eulogy, delivered in Budapest on October 19, 1996

I knew Erdős for 25 years, half my life, but still not very long compared to many people in this room. His memory was much better than mine; he often reminded me that we proved the theorems in our first paper in 1972 in a car as we drove back to Southern Illinois University in Carbondale after a meeting of the Illinois Number Theory Conference in Normal, Illinois. He visited me often in Carbondale, and even more often after I moved to New Jersey. He would frequently leave his winter coat in my house when he left for Europe in the spring, and retrieve it when he returned in the fall. I still have a carton of his belongings in my attic. My children Becky and Alex, who are five and seven years old, would ask, "When is Paul coming to visit again?" They liked his silly tricks for kids, like dropping a coin and catching it before it hit the floor. He was tolerant of the dietary rules in my house, which meant, for example, no milk in his espresso if we had just eaten meat.

He was tough. "No illegal thinking," he would say when we were working together. This meant no thinking about mathematical problems other than the ones we were working on at that time. In other words, he knew how to enforce party discipline.

Erdős loved to discuss politics, especially Sam and Joe, which, in his idiosyncratic language, meant the United States (Uncle Sam) and the Soviet Union (Joseph Stalin). His politics seemed to me to be the politics of the 30's, much to the left of my own. He embraced a kind of naive and altruistic socialism that I associate with idealistic intellectuals of his generation. He never wanted to believe what I told him about the Soviet Union as an "evil empire." I think he was genuinely saddened by the fact that the demise of communism in the Soviet Union meant the failure of certain dreams and principles that were important to him.

Erdős's cultural interests were narrowly focused. When he was in my house he always wanted to hear "noise" (that is, music), especially Bach. He loved to quote Hungarian poetry (in translation). I assume that when he was young he read literature (he was amazed that Anatole France is a forgotten literary figure today), but I don't think he read much anymore.

I subscribe to many political journals. When he came to my house he would look for the latest issue of *Foreign Affairs*, but usually disagreed with the contents. Not long ago, an American historian at Pacific Lutheran University published a book entitled *Ordinary Men*,[1] a study of how large numbers of "ordinary Germans," not just a few SS, actively and willingly participated in the murder of Jews. He found the book on my desk and read it, but didn't believe or didn't want to believe it could be true, because it conflicted with his belief in the natural goodness of ordinary men.

He had absolutely no interest in the visual arts. My wife was a curator at the Museum of Modern Art in New York, and we went with her one day to the museum. It has the finest collection of modern art in the world, but Paul was bored. After a few minutes, he went out to the scupture garden and started, as usual, to prove and conjecture.

[1]Christopher R. Browning, *Ordinary Men*, HarperCollins Publishers, New York, 1992.

Paul's mathematics was like his politics. He learned mathematics in the 1930's in Hungary and England, and England at that time was a kind of mathematical backwater. For the rest of his life he concentrated on the fields that he had learned as a boy. Elementary and analytic number theory, at the level of Landau, graph theory, set theory, probability theory, and classical analysis. In these fields he was an absolute master, a virtuoso.

At the same time, it is extraordinary to think of the parts of mathematics he never learned. Much of contemporary number theory, for example. In retrospect, probably the greatest number theorist of the 1930's was Hecke, but Erdős knew nothing about his work and cared less. Hardy and Littlewood dominated British number theory when Erdős lived in England, but I doubt they understood Hecke.

Traditions and knowledge need transmitters

There is an essay by Irving Segal[2] in the current issue of the *Bulletin of the American Mathematical Society*. He tells the story of the visit of another great Hungarian mathematician, John von Neumann, to Cambridge in the 1930's. After his lecture, Hardy remarked, "Obviously a very intelligent young man. But was that *mathematics*?"

Or Oxonians about phenomenology

A few months ago, on his last visit to New Jersey, I was telling Erdős something about p-adic analysis. Erdős was not interested. "You know," he said about the p-adic numbers, "they don't really exist."

Paul never learned algebraic number theory. He was offended – actually, he was furious – when André Weil wrote that analytic number theory is good mathematics, but analysis, not number theory.[3] Paul's "tit for tat" response was that André Weil did good mathematics, but it was algebra, not number theory. I think Paul was a bit shocked that a problem he did consider number theory, Fermat's Last Theorem, was solved using ideas and methods of Weil and other very sophisticated mathematicians.

It is idle to speculate about how great a mathematician Erdős was, as if one could put together a list of the top 10 or top 100 mathematicians of our century. His interests were broad, his conjectures, problems, and results profound, and his humanity extraordinary.

He was the "Bob Hope" of mathematics, a kind of vaudeville performer who told the same jokes and the same stories a thousand times. When he was scheduled to give yet another talk, no matter how tired he was, as soon as he was introduced to the audience, the adrenaline (or maybe amphetamine) would release into his system and he would bound onto the stage, full of energy, and do his routine for the 1001st time.

If he were here today, he would be sitting in the first row, half asleep, happy to be in the presence of so many colleagues, collaborators, and friends.

Yitgadal v'yitkadash sh'mei raba.

Y'hei zekronoh l'olam.

[2]Irving Segal, "*Noncommutative Geometry* by Alain Connes (book review)," *Bull. Amer. Math. Soc.* 33 (1996), 459–465

[3]Weil wrote, "...there is a subject in mathematics (it's a perfectly good and valid subject and it's perfectly good and valid mathematics) which is called Analytic Number Theory.... I would classify it under analysis...." (*Œuvres Scientifiques Collected Papers*, Springer-Verlag, New York, 1979, Volume III, p. 280).

May his memory be with us forever.[4]

2. Reconsideration

My brief talk at the Erdős conference was not intended for publication. Someone asked me for a copy, and it subsequently spread via e-mail. Many people who heard me in Budapest or who later read my eulogy told me that it helped them remember Paul as a human being, but others clearly disliked what I said. I confess I still don't know what disturbed them so deeply. It has less to do with Erdős, I think, than with the status of "Hungarian mathematics" in the scientific world.[5]

Everyone understands that Erdős was an extraordinary human being and a great mathematician who made major contributions to many parts of mathematics. He was a central figure in the creation of new fields, such as probabilistic number theory and random graphs. This part of the story is trivial.

It is also true, understood by almost everyone, and not controversial, that Erdős did not work in and never learned the central core of twentieth century mathematics. It is amazing to me how great were Erdos's contributions to mathematics, and how little he knew. He never learned, for example, the great discoveries in number theory that were made at the beginning of the twentieth century. These include, for example, Weil's work on diophantine equations, Artin's class field theory, and Hecke's monumental contributions to modular forms and analytic number theory. Erdős apparently knew nothing about Lie groups, Riemannian manifolds, algebraic geometry, algebraic topology, global analysis, or the deep ocean of mathematics connected with quantum mechanics and relativity theory. These subjects, already intensely investigated in the 1930's, were at the heart of twentieth century mathematics. How could a great mathematician not want to study these things?[6] This is the first Erdős paradox.

In the case of the Indian mathematician Ramanujan, whose knowledge was also deep but narrow, there is a discussion in the literature abut the possible sources of his mathematical education. The explanation of Hardy[7] and others is that the only serious book that was accessible to Ramanujan in India was Carr's *A Synopsis of Elementary Results in Pure and Applied Mathematics*, and that Ramanujan lacked a broad mathematical culture because he did not have access to books and journals in India. But Hungary was not India; there were libraries, books, and journals in Budapest, and in other places where Erdős lived in the 1930's and 1940's.

For the past half century, "Hungarian mathematics" has been a term of art to describe the kind of mathematics that Erdős did.[8] It includes combinatorics,

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[4]I ended my eulogy with a sentence in Aramaic and a sentence in Hebrew. The first is the first line of the Kaddish, the Jewish prayer for the dead. Immediately following the second sentence is its English translation.

[5]cf. L. Babai, "In and out of Hungary: Paul Erdős, his friends, and times," in: *Combinatorics, Paul Erdős is Eighty (Volume 2), Keszthely (Hungary) 1993*, Bolyai Society Mathematical Studies, Budapest, 1996, pp. 7–95.

[6]This suggests the fundamental question: How much, or how little, must one know in order to do great mathematics?

[7]"It was a book of a very different kind, Carr's *Synopsis*, which first aroused Ramanujan's full powers," according to G. H. Hardy, in his book *Ramanujan*, Chelsea Publishing, New York, 1959, p. 2

[8]For example, Joel Spencer, "I felt . . . I was working on 'Hungarian mathematics'," quoted in Babai, *op. cit.*

graph theory, combinatorial set theory, and elementary and combinatorial number theory. Not all Hungarians do this kind of mathematics, of course, and many non-Hungarians do Hungarian mathematics. It happens that combinatorial reasoning is central to theoretical computer science, and "Hungarian mathematics" commands vast respect in the computer science world. It is also true, however, that for many years combinatorics did not have the highest reputation among mathematicians in the ruling subset of the research community, exactly because combinatorics was concerned largely with questions that they believed (incorrectly) were not central to twentieth century mathematics.[9]

In a volume in honor of Erdős's 70th birthday, Ernst Straus wrote, "In our century, in which mathematics is so strongly dominated by 'theory constructors' [Erdős] has remained the prince of problem solvers and the absolute monarch of problem posers."[10] I disagree. There is, as Gel'fand often said, only one mathematics. There is no separation of mathematics into "theory" and "problems." But there is an interesting lurking issue.

In his lifetime, did Erdős get the recognition he deserved? Even though Erdős received almost every honor that can be given to a mathematician, some of his friends believe that he was still insufficiently appreciated, and they are bitter on his behalf. He was awarded a Wolf Prize and a Cole Prize, but he did not get a Fields Medal or a permanent professorship at the Institute for Advanced Study. He traveled from one university to another across the United States, and was never without an invitation to lecture somewhere, but his mathematics was not highly regarded by the power brokers of mathematics. To them, his methods were insufficiently abstruse and obscure; they did not require complicated machinery. Paul invented diabolically clever arguments from arithmetic, combinatorics, and probability to solve problems. But the technique was too simple, too elementary. It was suspicious. The work could not be "deep."

None of this seemed to matter to Erdős, who was content to prove and conjecture and publish more than 1,500 papers.

Not because of politicking, but because of computer science and because his mathematics was always beautiful, in the past decade the reputation of Erdős and the respect paid to discrete mathematics have increased exponentially. The *Annals of Mathematics* will now publish papers in combinatorics, and the most active seminar at the Institute for Advanced Study is in discrete mathematics and theoretical computer science. Fields Medals are awarded to mathematicians who solve Erdős-type problems. Science has changed.

In 1988 Alexander Grothendieck was awarded the Crafoord Prize of the Swedish Academy of Sciences. In the letter to the Swedish Academy in which he declined the prize, he wrote, "Je suis persuadé que la seule épreuve décisive pour la fécundité d'idées ou d'une vision nouvelles est celle du temps. La fécondité se reconnait par la progéniture, et non par les honneurs."[11]

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[9] For example, S. Mac Lane criticized "emphasizing too much of a Hungarian view of mathematics," in: "The health of mathematics," *Math.Intelligencer* 5 (1983), 53–55.

[10] E. G. Straus, "Paul Erdős at 70," *Combinatorica* 3 (1983), 245–246. Tim Gowers revisited this notion in his essay, "The two cultures of mathematics," published in *Mathematics: Frontiers and Perspectives*, American Mathematical Society, 2000.

[11] "I believe that time gives the only definite proof of the fertility of new ideas or a new vision. We recognize fertility by its offspring, and not by honors."

Time has proved the fertility and richness of Erdős's work. The second Erdős paradox is that his methods and results, considered marginal in the twentieth century, have become central in twenty-first century mathematics.

May his memory be with us forever.

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