The Meaning of Einstein's Equation

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Abstract

This is a brief introduction to general relativity, designed for both students and teachers of the subject. While there are many excellent expositions of general relativity, few adequately explain the geometrical meaning of the basic equation of the theory: Einstein's equation. Here we give a simple formulation of this equation in terms of the motion of freely falling test particles. We also sketch some of the consequences of this formulation and explain how it is equivalent to the usual one in terms of tensors. Finally, we include an annotated bibliography of books, articles and websites suitable for the student of relativity.

1 Introduction

General relativity explains gravity as the curvature of spacetime. It's all about geometry. The basic equation of general relativity is called Einstein's equation. In units where $c = 8\pi G = 1$, it says

$$G_{\alpha\beta} = T_{\alpha\beta}. \tag{1}$$

It looks simple, but what does it mean? Unfortunately, the beautiful geometrical meaning of this equation is a bit hard to find in most treatments of relativity. There are many nice popularizations that explain the philosophy behind relativity and the idea of curved spacetime, but most of them don't get around to explaining Einstein's equation and showing how to work out its consequences. There are also more technical introductions which explain Einstein's equation in detail — but here the geometry is often hidden under piles of tensor calculus.

This is a pity, because in fact there is an easy way to express the whole content of Einstein's equation in plain English. In fact, after a suitable prelude, one can summarize it in a single sentence! One needs a lot of mathematics to derive all the consequences of this sentence, but it is still worth seeing — and we can work out *some* of its consequences quite easily.

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In what follows, we start by outlining some differences between special and general relativity. Next we give a verbal formulation of Einstein's equation. Then we derive a few of its consequences concerning tidal forces, gravitational waves, gravitational collapse, and the big bang cosmology. In the last section we explain why our verbal formulation is equivalent to the usual one in terms of tensors. This article is mainly aimed at those who teach relativity, but except for the last section, we have tried to make it accessible to students, as a sketch of how the subject might be introduced. We conclude with a bibliography of sources to help teach the subject.

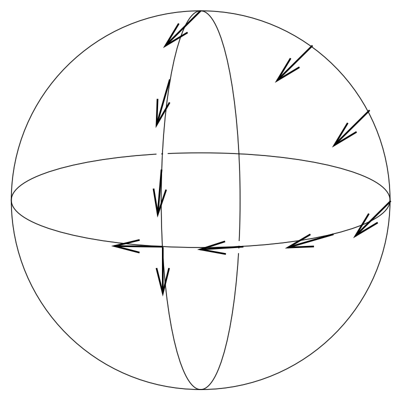
2 Preliminaries

Before stating Einstein's equation, we need a little preparation. We assume the reader is somewhat familiar with special relativity — otherwise general relativity will be too hard. But there are some big differences between special and general relativity, which can cause immense confusion if neglected.

In special relativity, we cannot talk about *absolute* velocities, but only *relative* velocities. For example, we cannot sensibly ask if a particle is at rest, only whether it is at rest relative to another. The reason is that in this theory, velocities are described as vectors in 4-dimensional spacetime. Switching to a different inertial coordinate system can change which way these vectors point relative to our coordinate axes, but not whether two of them point the same way.

In general relativity, we cannot even talk about *relative* velocities, except for two particles at the same point of spacetime — that is, at the same place at the same instant. The reason is that in general relativity, we take very seriously the notion that a vector is a little arrow sitting at a particular point in spacetime. To compare vectors at different points of spacetime, we must carry one over to the other. The process of carrying a vector along a path without turning or stretching it is called 'parallel transport'. When spacetime is curved, the result of parallel transport from one point to another depends on the path taken! In fact, this is the very definition of what it means for spacetime to be curved. Thus it is ambiguous to ask whether two particles have the same velocity vector unless they are at the same point of spacetime.

It is hard to imagine the curvature of 4-dimensional spacetime, but it is easy to see it in a 2-dimensional surface, like a sphere. The sphere fits nicely in 3-dimensional flat Euclidean space, so we can visualize vectors on the sphere as 'tangent vectors'. If we parallel transport a tangent vector from the north pole to the equator by going straight down a meridian, we get a different result than if we go down another meridian and then along the equator:



Because of this analogy, in general relativity vectors are usually called 'tangent vectors'. However, it is important not to take this analogy too seriously. Our curved spacetime need not be embedded in some higher-dimensional flat spacetime for us to understand its curvature, or the concept of tangent vector. The mathematics of tensor calculus is designed to let us handle these concepts 'intrinsically' — i.e., working solely within the 4-dimensional spacetime in which we find ourselves. This is one reason tensor calculus is so important in general relativity.

Now, in special relativity we can think of an inertial coordinate system, or 'inertial frame', as being defined by a field of clocks, all at rest relative to each other. In general relativity this makes no sense, since we can only unambiguously define the relative velocity of two clocks if they are at the same location. Thus the concept of inertial frame, so important in special relativity, is *banned* from general relativity!

If we are willing to put up with limited accuracy, we can still talk about the relative velocity of two particles in the limit where they are very close, since curvature effects will then be very small. In this approximate sense, we can talk about a 'local' inertial coordinate system. However, we must remember that this notion makes perfect sense only in the limit where the region of spacetime covered by the coordinate system goes to zero in size.

Einstein's equation can be expressed as a statement about the relative acceleration of very close test particles in free fall. Let us clarify these terms a bit. A 'test particle' is an idealized point particle with energy and momentum so small that its effects on spacetime curvature are negligible. A particle is said to be in 'free fall' when its motion is affected by no forces except gravity. In general relativity, a test particle in free fall will trace out a 'geodesic'. This means that its velocity vector is parallel transported along the curve it traces out in spacetime. A geodesic is the closest thing there is to a straight line in curved spacetime.

Again, all this is easier to visualize in 2d space rather than 4d spacetime. A person walking on a sphere 'following their nose' will trace out a geodesic — that is, a great circle. Suppose two people stand side-by-side on the equator and start walking north, both following geodesics. Though they start out walking parallel to each other, the distance between them will gradually start to shrink, until finally they bump into each other at the north pole. If they didn't understand the curved geometry of the sphere, they might think a 'force' was pulling them together.

Similarly, in general relativity gravity is not really a 'force', but just a manifestation of the curvature of spacetime. Note: not the curvature of space, but of *spacetime*. The distinction is crucial. If you toss a ball, it follows a parabolic path. This is far from being a geodesic in *space*: space is curved by the Earth's gravitational field, but it is certainly not so curved as all that! The point is that while the ball moves a short distance in space, it moves an enormous distance in *time*, since one second equals about 300,000 kilometers in units where $c = 1$. This allows a slight amount of spacetime curvature to have a noticeable effect.

3 Einstein's Equation

To state Einstein's equation in simple English, we need to consider a round ball of test particles that are all initially at rest relative to each other. As we have seen, this is a sensible notion only in the limit where the ball is very small. If we start with such a ball of particles, it will, to second order in time, become an ellipsoid as time passes. This should not be too surprising, because any linear transformation applied to a ball gives an ellipsoid, and as the saying goes, "everything is linear to first order". Here we get a bit more: the relative velocity of the particles starts out being zero, so to first order in time the ball does not change shape at all: the change is a second-order effect.

Let $V(t)$ be the volume of the ball after a proper time t has elapsed, as measured by the particle at the center of the ball. Then Einstein's equation says:

$$\frac{\ddot{V}}{V}\Big|_{t=0} = -\frac{1}{2}\left(\begin{array}{l} \text{flow of } t\text{−momentum in } t \text{ direction } + \\ \text{flow of } x\text{−momentum in } x \text{ direction } + \\ \text{flow of } y\text{−momentum in } y \text{ direction } + \\ \text{flow of } z\text{−momentum in } z \text{ direction} \end{array}\right)$$

where these flows are measured at the center of the ball at time zero, using local inertial coordinates. These flows are the diagonal components of a 4×4 matrix T called the 'stress-energy tensor'. The components $T_{\alpha\beta}$ of this matrix say how much momentum in the α direction is flowing in the β direction through a given point of spacetime, where $\alpha, \beta = t, x, y, z$. The flow of t-momentum in the t-direction is just the energy density, often denoted ρ. The flow of x-momentum in the x-direction is the 'pressure in the x direction' denoted P_x, and similarly for y and z. It takes a while to figure out why pressure is really the flow of momentum, but it is eminently worth doing. Most texts explain this fact by considering the example of an ideal gas.

In any event, we may summarize Einstein's equation as follows:

$$\frac{\ddot{V}}{V}\Big|_{t=0} = -\frac{1}{2}(\rho + P_x + P_y + P_z). \qquad (2)$$

This equation says that positive energy density and positive pressure curve spacetime in a way that makes a freely falling ball of point particles tend to shrink. Since $E = mc^2$ and we are working in units where $c = 1$, ordinary mass density counts as a form of energy density. Thus a massive object will make a swarm of freely falling particles at rest around it start to shrink. In short: *gravity attracts.*

We promised to state Einstein's equation in plain English, but have not done so yet. Here it is:

> Given a small ball of freely falling test particles initially at rest with respect to each other, the rate at which it begins to shrink is proportional to its volume times: the energy density at the center of the ball, plus the pressure in the x direction at that point, plus the pressure in the y direction, plus the pressure in the z direction.

One way to prove this is by using the Raychaudhuri equation, discussions of which can be found in the textbooks by Wald and by Ciufolini and Wheeler cited in the bibliography. But an elementary proof can also be given starting from first principles, as we will show in the final section of this article.

The reader who already knows some general relativity may be somewhat skeptical that all of Einstein's equation is encapsulated in this formulation. After all, Einstein's equation in its usual tensorial form is really a bunch of equations: the left and right sides of equation (1) are 4×4 matrices. It is hard to believe that the single equation (2) captures all that information. It does, though, as long as we include one bit of fine print: in order to get the full content of the Einstein equation from equation (2), we must consider small balls with *all possible* initial velocities — i.e., balls that begin at rest in all possible local inertial reference frames.

Before we begin, it is worth noting an even simpler formulation of Einstein's equation that applies when the pressure is the same in every direction:

> Given a small ball of freely falling test particles initially at rest with respect to each other, the rate at which it begins to shrink is proportional to its volume times: the energy density at the center of the ball plus three times the pressure at that point.

This version is only sufficient for 'isotropic' situations: that is, those in which all directions look the same in some local inertial reference frame. But, since the simplest models of cosmology treat the universe as isotropic — at least approximately, on large enough distance scales — this is all we shall need to derive an equation describing the big bang!

4 Some Consequences

The formulation of Einstein's equation we have given is certainly not the best for most applications of general relativity. For example, in 1915 Einstein used general relativity to correctly compute the anomalous precession of the orbit of Mercury and also the deflection of starlight by the Sun's gravitational field. Both these calculations would be very hard starting from equation (2); they really call for the full apparatus of tensor calculus.

However, we can easily use our formulation of Einstein's equation to get a qualitative — and sometimes even quantitative — understanding of *some* consequences of general relativity. We have already seen that it explains how gravity attracts. We sketch a few other consequences below. These include Newton's inverse-square force law, which holds in the limit of weak gravitational fields and small velocities, and also the equations governing the big bang cosmology.

Tidal Forces, Gravitational Waves

We begin with some qualitative consequences of Einstein's equation. Let $V(t)$ be the volume of a small ball of test particles in free fall that are initially at rest relative to each other. In the vacuum there is no energy density or pressure, so $\ddot{V}|_{t=0} = 0$, but the curvature of spacetime can still distort the ball. For example, suppose you drop a small ball of instant coffee when making coffee in the morning. The grains of coffee closer to the earth accelerate towards it a bit more, causing the ball to start stretching in the vertical direction. However, as the grains all accelerate towards the center of the earth, the ball also starts being squashed in the two horizontal directions. Einstein's equation says that if we treat the coffee grains as test particles, these two effects cancel each other when we calculate the second derivative of the ball's volume, leaving us with $\ddot{V}|_{t=0} = 0$. It is a fun exercise to check this using Newton's theory of gravity!

This stretching/squashing of a ball of falling coffee grains is an example of what people call 'tidal forces'. As the name suggests, another example is the tendency for the ocean to be stretched in one direction and squashed in the other two by the gravitational pull of the moon.

Gravitational waves are another example of how spacetime can be curved even in the vacuum. General relativity predicts that when any heavy object wiggles, it sends out ripples of spacetime curvature which propagate at the speed of light. This is far from obvious starting from our formulation of Einstein's equation! It also predicts that as one of these ripples of curvature passes by, our small ball of initially test particles will be stretched in one transverse direction while being squashed in the other transverse direction. From what we have already said, these effects must precisely cancel when we compute $\ddot{V}|_{t=0}$.

Hulse and Taylor won the Nobel prize in 1993 for careful observations of a binary neutron star which is slowly spiraling down, just as general relativity predicts it should, as it loses energy by emitting gravitational radiation. Gravitational waves have not been *directly* observed, but there are a number of projects underway to detect them. For example, the LIGO project will bounce a laser between hanging mirrors in an L-shaped detector, to see how one leg of the detector is stretched while the other is squashed. Both legs are 4 kilometers long, and the detector is designed to be sensitive to a 10^{-18}-meter change in length of the arms.

Gravitational Collapse

Another remarkable feature of Einstein's equation is the pressure term: it says that not only energy density but also pressure causes gravitational attraction. This may seem to violate our intuition that pressure makes matter want to expand! Here, however, we are talking about *gravitational* effects of pressure, which are undetectably small in everyday circumstances. To see this, let's restore the factors of c. Also, let's remember that in ordinary circumstances most of the energy is in the form of rest energy, so we can write the energy density ρ as $\rho_m c^2$, where ρ_m is the ordinary mass density:

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -4\pi G(\rho_m + \frac{1}{c^2}(P_x + P_y + P_z)).$$

On the human scale all of the terms on the right are small, since G is very small. (Gravity is a weak force!) Furthermore, the pressure terms are much smaller than the mass density term, since they are divided by an extra factor of c^2.

There are a number of important situations in which ρ does not dominate over P. In a neutron star, for example, which is held up by the degeneracy pressure of the neutronium it consists of, pressure and energy density contribute comparably to the right-hand side of Einstein's equation. Moreover, above a mass of about 2 solar masses a nonrotating neutron star will inevitably collapse to form a black hole, thanks in part to the gravitational attraction caused by pressure. In fact, any object of mass M will form a black hole if it is compressed to a radius smaller than its Schwarzschild radius, $R = 2GM/c^2$.

Newton's Inverse-Square Force Law

A basic test of general relativity is to check that it reduces to good old Newtonian gravity in the limit where gravitational effects are weak and velocities are small compared to the speed of light. To do this, we can use our formulation of Einstein's equation to derive Newton's inverse-square force law for a planet with mass M and radius R. Since we can only do this when gravitational effects are weak, we must assume that the planet's radius is much greater than its Schwarzschild radius: $R \gg M$ in units where $c = 8\pi G = 1$. Then the curvature of *space* – as opposed to spacetime – is small. To keep things simple, we make a couple of additional assumptions: the planet has uniform density ρ, and the pressure is negligible.

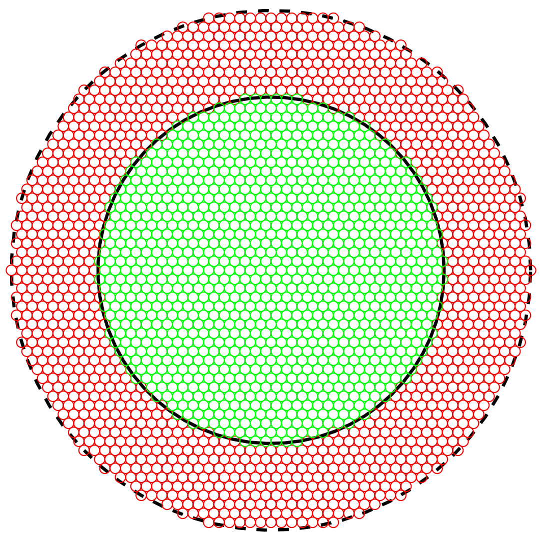
We want to derive the familiar Newtonian result

$$a = -\frac{GM}{r^2}$$

giving the radial gravitational acceleration a of a test particle at distance r from the planet's center, with $r > R$ of course.

To do this, let S be a sphere of radius r centered on the planet. Fill the interior of the sphere with test particles, all of which are initially at rest relative to the planet. At first, this might seem like an illegal thing to do: we know that notions like 'at rest' only make sense in an infinitesimal neighborhood, and r is not infinitesimal. But because space is nearly flat for weak gravitational fields, we can get away with this.

We can't apply our formulation of Einstein's equation directly to S, but we can apply it to any infinitesimal sphere within S. In the picture below, the solid black circle represents the planet, and the dashed circle is S. The interior of S has been divided up into many tiny spheres filled with test particles. Green spheres are initially inside the planet, and red spheres are outside.



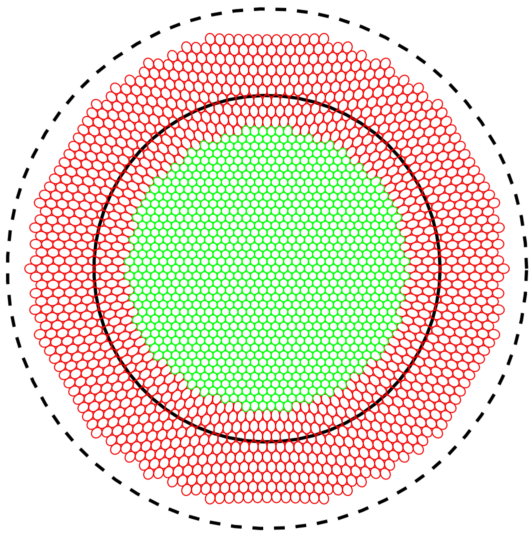
Suppose we pick a tiny green sphere that lies within the planet's volume, a distance less than R from the center. Our formulation of Einstein's equation tells us that the fractional change in volume of this sphere will be

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -\frac{1}{2}\rho \qquad \text{(inside the planet).}$$

On the other hand, as we saw in our discussion of tidal effects, spheres outside the planet's volume will be distorted in shape by tidal effects, but remain unchanged in volume. So, any little red sphere that lies outside the planet will undergo no change in volume at all:

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = 0 \qquad \text{(outside the planet).}$$

Thus, after a short time δt has elapsed, the test particles will be distributed like this:



Because the volume occupied by the spheres of particles within the planet went down, the whole big sphere of test particles had to shrink. Now consider the following question: what is the change in volume of the large sphere of test particles? All the little green spheres shrank by the same fractional amount:

$$\frac{\delta V}{V} = \frac{1}{2}\left(\frac{\ddot{V}}{V}\right)(\delta t)^2 = -\frac{1}{4}\rho(\delta t)^2,$$

to leading order in δt, while all the little red spheres underwent no change in volume. So, if V_P stands for the volume occupied by the test particles that were initially inside the planet, and V_S stands for the volume of the big sphere S, then the overall change in volume is

$$\begin{aligned}\delta V_S &= \delta V_P \\ &= \left(\frac{\delta V}{V}\right) V_P \\ &= -\frac{1}{4}\rho(\delta t)^2\, V_P \\ &= -\frac{1}{4}M(\delta t)^2.\end{aligned}$$

We can also express δV_S in terms of the change in the sphere's radius, δr:

$$\delta V_S = 4\pi r^2 \delta r.$$

Setting the two expressions for δV_S equal, we see

$$\delta r = -\frac{M}{16\pi r^2}(\delta t)^2.$$

But the change in radius δr is just the radial displacement of a test particle on the outer edge of the sphere, so $\delta r = \frac{1}{2}a(\delta t)^2$ where a is the radial acceleration of this particle. Plugging this into the above equation, we obtain

$$a = -\frac{M}{8\pi r^2}.$$

We've been working all along in units in which $8\pi G = 1$. Switching back to conventional units, we obtain

$$a = -\frac{GM}{r^2}$$

just as Newton said.

The Big Bang

We can also derive some basic facts about the big bang cosmology. Let us assume the universe is not only expanding but also homogeneous and isotropic. The expansion of the universe is vouched for by the redshifts of distant galaxies. The other assumptions also seem to be approximately correct, at least when we average over small-scale inhomogeneities such as stars and galaxies. For simplicity, we will imagine the universe is homogeneous and isotropic even on small scales.

An observer at any point in such a universe would see all objects receding from her. Suppose that, at some time $t = 0$, she identifies a small ball B of test particles centered on her. Suppose this ball expands with the universe, remaining spherical as time passes because the universe is isotropic. Let $R(t)$ stand for the radius of this ball as a function of time. The Einstein equation will give us an equation of motion for $R(t)$. In other words, it will say how the expansion rate of the universe changes with time.

It is tempting to apply equation (2) to the ball B, but we must take care. This equation applies to a ball of particles that are initially at rest relative to one another — that is, one whose radius is not changing at $t = 0$. However, the ball B is expanding at $t = 0$. Thus, to apply our formulation of Einstein's equation, we must introduce a second small ball of test particles that are at rest relative to each other at $t = 0$.

Let us call this second ball B', and call its radius as a function of time $r(t)$. Since the particles in this ball begin at rest relative to one another, we have

$$\dot{r}(0) = 0.$$

To keep things simple, let us also assume that at $t = 0$ both balls have the exact same size:

$$r(0) = R(0).$$

Equation (2) applies to the ball B', since the particles in *this* ball are initially at rest relative to each other. Since the volume of this ball is proportional to r^3, and since $\dot{r} = 0$ at $t = 0$, the left-hand side of equation (2) is simply

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = \left.\frac{3\ddot{r}}{r}\right|_{t=0}.$$

Since we are assuming the universe is isotropic, we know that the various components of pressure are equal: $P_x = P_y = P_z = P$. Einstein's equation (2) thus says that

$$\left.\frac{3\ddot{r}}{r}\right|_{t=0} = -\frac{1}{2}(\rho + 3P).$$

We would much prefer to rewrite this expression in terms of R rather than r. Fortunately, we can do this. At $t = 0$, the two spheres have the same radius: $r(0) = R(0)$. Furthermore, the second derivatives are the same: $\ddot{r}(0) = \ddot{R}(0)$. This follows from the equivalence principle, which says that, at any given location, particles in free fall do not accelerate with respect to each other. At the moment $t = 0$, each test particle on the surface of the ball B is right next to a corresponding test particle in B'. Since they are not accelerating with respect to each other, the observer at the origin must see both particles accelerating in the same way, so $\ddot{r}(0) = \ddot{R}(0)$. It follows that we can replace r with R in the above equation, obtaining

$$\left.\frac{3\ddot{R}}{R}\right|_{t=0} = -\frac{1}{2}(\rho + 3P).$$

We derived this equation for a very small ball, but in fact it applies to a ball of any size. This is because, in a homogeneous expanding universe, the balls of all radii must be expanding at the same fractional rate. In other words, $\ddot{R}/R$ is independent of the radius R, although it can depend on time. Also, there is nothing special in this equation about the moment $t = 0$, so the equation must apply at all times. In summary, therefore, the basic equation describing the big bang cosmology is

$$\frac{3\ddot{R}}{R} = -\frac{1}{2}(\rho + 3P), \tag{3}$$

where the density ρ and pressure P can depend on time but not on position. Here we can imagine R to be the separation between any two 'galaxies'.

To go further, we must make more assumptions about the nature of the matter filling the universe. One simple model is a universe filled with pressureless matter. Until recently, this was thought to be an accurate model of our universe. Setting $P = 0$, we obtain

$$\frac{3\ddot{R}}{R} = -\frac{\rho}{2}.$$

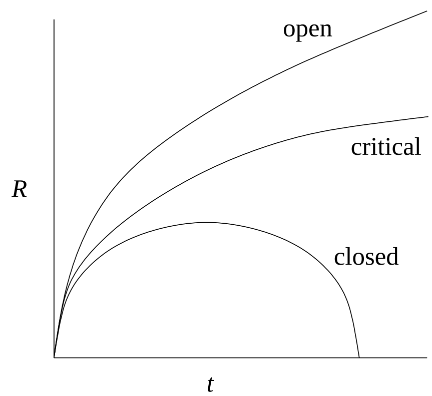
If the energy density of the universe is mainly due to the mass in galaxies, 'conservation of galaxies' implies that $\rho R^3 = k$ for some constant k. This gives

$$\frac{3\ddot{R}}{R} = -\frac{k}{2R^3}$$

or

$$\ddot{R} = -\frac{k}{6R^2}.$$

Amusingly, this is the same as the equation of motion for a particle in an attractive $1/R^2$ force field. In other words, the equation governing this simplified cosmology is the same as the Newtonian equation for what happens when you throw a ball vertically upwards from the earth! This is a nice example of the unity of physics. Since "whatever goes up must come down — unless it exceeds escape velocity," the solutions of this equation look roughly like this:



So, the universe started out with a big bang! It will expand forever if its current rate of expansion is sufficiently high compared to its current density, but it will recollapse in a 'big crunch' otherwise.

The Cosmological Constant

The simplified big bang model just described is inaccurate for the very early history of the universe, when the pressure of radiation was important. Moreover, recent observations seem to indicate that it is seriously inaccurate even in the present epoch. First of all, it seems that much of the energy density is not accounted for by known forms of matter. Still more shocking, it seems that the expansion of the universe may be accelerating rather than slowing down! One possibility is that the energy density and pressure are nonzero even for the vacuum. For the vacuum to not pick out a preferred notion of 'rest', its stress-energy tensor must be proportional to the metric. In local inertial coordinates this means that the stress-energy tensor of the vacuum must be

$$T = \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & -\Lambda \end{pmatrix}$$

where Λ is called the 'cosmological constant'. This amounts to giving empty space an energy density equal to Λ and pressure equal to $-\Lambda$, so that $\rho + 3P$ for the vacuum is -2Λ. Here pressure effects dominate because there are more dimensions of space than of time! If we add this cosmological constant term to equation (3), we get

$$\frac{3\ddot{R}}{R} = -\frac{1}{2}(\rho + 3P - 2\Lambda), \tag{4}$$

where ρ and P are the energy density and pressure due to matter. If we treat matter as we did before, this gives

$$\frac{3\ddot{R}}{R} = -\frac{k}{2R^3} + \Lambda.$$

Thus, once the universe expands sufficiently, the cosmological constant becomes more important than the energy density of matter in determining the fate of the universe. If $\Lambda > 0$, a roughly exponential expansion will then ensue. This seems to be happening in our universe now.

Spatial Curvature

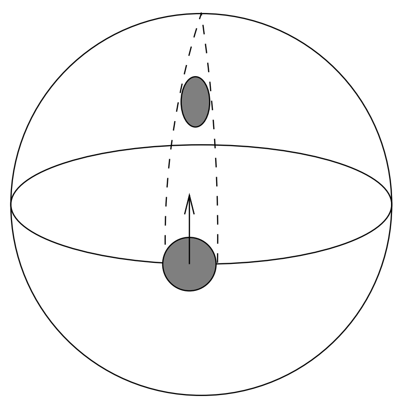
We have emphasized that gravity is due not just to the curvature of space, but of *spacetime*. In our verbal formulation of Einstein's equation, this shows up in the fact that we consider particles moving forwards in time and study how their paths deviate in the space directions. However, Einstein's equation also gives information about the curvature of space. To illustrate this, it is easiest to consider not an expanding universe but a static one.

When Einstein first tried to use general relativity to construct a model of the entire universe, he assumed that the universe must be static — although he is said to have later described this as "his greatest blunder". As we did in the previous section, Einstein considered a universe containing ordinary matter with density ρ, no pressure, and a cosmological constant Λ. Such a universe can be static — the galaxies can remain at rest with respect to each other — only if the right-hand side of equation (4) is zero. In such a universe, the cosmological constant and the density must be carefully 'tuned' so that $\rho = 2\Lambda$. It is tempting to conclude that spacetime in this model is just the good old flat Minkowski spacetime of special relativity. In other words, one might guess that there are no gravitational effects at all. After all, the right-hand side of Einstein's equation was tuned to be zero. This would be a mistake, however. It is instructive to see why.

Remember that equation (2) contains all the information in Einstein's equation only if we consider all possible small balls. In all of the cosmological applications so far, we have applied the equation only to balls whose centers were at rest with respect to the local matter. It turns out that only for such balls is the right-hand side of equation (2) zero in the Einstein static universe.

To see this, consider a small ball of test particles, initially at rest relative to each other, that is moving with respect to the matter in the universe. In the local rest frame of such a ball, the right-hand side of equation (2) is nonzero. For one thing, the pressure due to the matter no longer vanishes. Remember that pressure is the flux of momentum. In the frame of our moving sphere, matter is flowing by. Also, the energy density goes up, both because the matter has kinetic energy in this frame and because of Lorentz contraction. The end result, as the reader can verify, is that the right-hand side of equation (2) is negative for such a moving sphere. In short, although a stationary ball of test particles remains unchanged in the Einstein static universe, our moving ball shrinks!

This has a nice geometric interpretation: the geometry in this model has spatial curvature. As we noted in section 2, on a positively curved surface such as a sphere, initially parallel lines converge towards one another. The same thing happens in the three-dimensional space of the Einstein static universe. In fact, the geometry of space in this model is that of a 3-sphere. This picture illustrates what happens:



One dimension is suppressed in this picture, so the two-dimensional spherical surface shown represents the three-dimensional universe. The small shaded circle on the surface represents our tiny sphere of test particles, which starts at the equator and moves north. The sides of the sphere approach each other along the dashed geodesics, so the sphere shrinks in the transverse direction, although its diameter in the direction of motion does not change.

As an exercise, the reader who wants to test his understanding can fill in the mathematical details in this picture and determine the radius of the Einstein static universe in terms of the density. Here are step-by-step instructions:

- Imagine an observer moving at speed v through a cloud of stationary particles of density ρ. Use special relativity to determine the energy density and pressure in the observer's rest frame. Assume for simplicity that the observer is moving fairly slowly, so keep only the lowest-order nonvanishing term in a power series in v.

- Apply equation (2) to a sphere in this frame, including the contribution due to the cosmological constant (which is the same in all reference frames). You should find that the volume of the sphere decreases with $\ddot{V}/V \propto -\rho v^2$ to leading order in v.

- Suppose that space in this universe has the geometry of a large 3-sphere of radius R_U. Show that the radii in the directions transverse to the motion start to shrink at a rate given by $(\ddot{R}/R)\Big|_{t=0} = -v^2/R_U^2$. (If, like most people, you are better at visualizing 2-spheres than 3-spheres, do this step by considering a small circle moving on a 2-sphere, as shown above, rather than a small sphere moving on a 3-sphere. The result is the same.)

- Since our little sphere is shrinking in two dimensions, its volume changes at a rate $\ddot{V}/V = 2\ddot{R}/R$. Use Einstein's equation to relate the radius R_U of the universe to the density ρ.

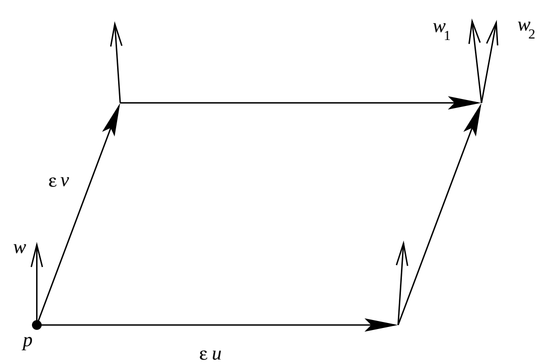
The final answer is $R_U = \sqrt{2/\rho}$. As you can confirm in standard textbooks, this is correct.

Spatial curvature like this shows up in the expanding cosmological models described earlier in this section as well. In principle, the curvature radius can be found from our formulation of Einstein's equation by similar reasoning in these expanding models. In fact, however, such a calculation is extremely messy. Here the apparatus of tensor calculus comes to our rescue.

5 The Mathematical Details

To see why equation (2) is equivalent to the usual formulation of Einstein's equation, we need a bit of tensor calculus. In particular, we need to understand the Riemann curvature tensor and the geodesic deviation equation. For a detailed explanation of these, the reader must turn to some of the texts in the bibliography. Here we briefly sketch the main ideas.

When spacetime is curved, the result of parallel transport depends on the path taken. To quantify this notion, pick two vectors u and v at a point p in spacetime. In the limit where $\epsilon \to 0$, we can approximately speak of a 'parallelogram' with sides ϵu and ϵv. Take another vector w at p and parallel transport it first along ϵv and then along ϵu to the opposite corner of this parallelogram. The result is some vector w_1. Alternatively, parallel transport w first along ϵu and then along ϵv. The result is a slightly different vector, w_2:



The limit

$$\lim_{\epsilon \to 0} \frac{w_2 - w_1}{\epsilon^2} = R(u, v)w \tag{5}$$

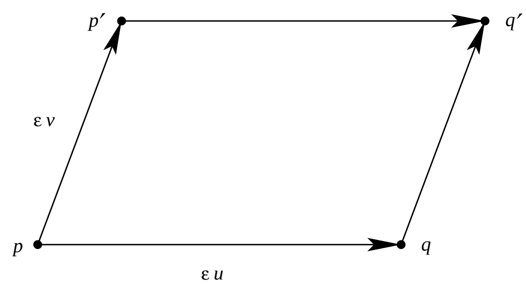
is well-defined, and it measures the curvature of spacetime at the point p. In local coordinates we can write it as

$$R(u, v)w = R^{\alpha}_{\beta\gamma\delta} u^{\beta} v^{\gamma} w^{\delta},$$

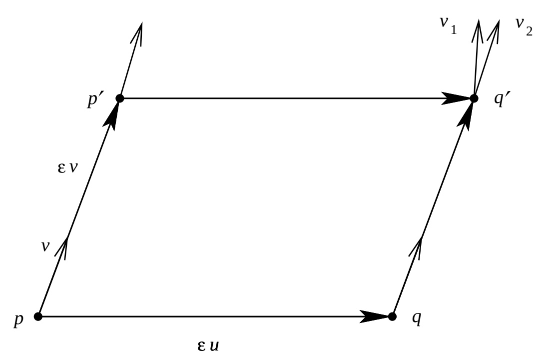
where as usual we sum over repeated indices. The quantity $R^{\alpha}_{\beta\gamma\delta}$ is called the 'Riemann curvature tensor'.

We can use this tensor to compute the relative acceleration of nearby particles in free fall if they are initially at rest relative to one another. Consider two freely falling particles at nearby points p and q. Let v be the velocity of the particle at p, and let ϵu be the vector from p to q. Since the two particles start out at rest relative to one other, the velocity of the particle at q is obtained by parallel transporting v along ϵu.

Now let us wait a short while. Both particles trace out geodesics as time passes, and at time ϵ they will be at new points, say p' and q'. The point p' is displaced from p by an amount ϵv, so we get a little parallelogram, exactly as in the definition of the Riemann curvature:



Next let us compute the new relative velocity of the two particles. To compare vectors we must carry one to another using parallel transport. Let v_1 be the vector we get by taking the velocity vector of the particle at p' and parallel transporting it to q' along the top edge of our parallelogram. Let v_2 be the velocity of the particle at q'. The difference $v_2 - v_1$ is the new relative velocity. Here is a picture of the whole situation:



The vector v is depicted as shorter than ϵv for purely artistic reasons.

It follows that over this passage of time, the average relative acceleration of the two particles is $a = (v_2 - v_1)/\epsilon$. By equation (5),

$$\lim_{\epsilon \to 0} \frac{v_2 - v_1}{\epsilon^2} = R(u,v)v,$$

so

$$\lim_{\epsilon \to 0} \frac{a}{\epsilon} = R(u,v)v.$$

This is called the 'geodesic deviation equation'. From the definition of the Riemann curvature it is easy to see that $R(u,v)w = -R(v,u)w$, so we can also write this equation as

$$\lim_{\epsilon \to 0} \frac{a^\alpha}{\epsilon} = -R^\alpha_{\beta\gamma\delta} v^\beta u^\gamma v^\delta. \tag{6}$$

Using this equation we can work out the second time derivative of the volume $V(t)$ of a small ball of test particles that start out at rest relative to each other. As we mentioned earlier, to second order in time the ball changes to an ellipsoid. Furthermore, since the ball starts out at rest, the principal axes of this ellipsoid don't rotate initially. We can therefore adopt local inertial coordinates in which, to second order in t, the center of the ball is at rest and the three principal axes of the ellipsoid are aligned with the three spatial coordinates. Let $r^j(t)$ represent the radius of the jth axis of the ellipsoid as a function of time. If the ball's initial radius is ϵ, then

$$r^j(t) = \epsilon + \frac{1}{2} a^j t^2 + O(t^3),$$

or in other words,

$$\lim_{t \to 0} \frac{\ddot{r}^j}{r^j} = \lim_{t \to 0} \frac{a^j}{\epsilon}.$$

Here the acceleration a^j is given by equation (6), with u being a unit vector in the jth coordinate direction and v being the velocity of the ball, which is a unit vector in the time direction. In other words,

$$\lim_{t \to 0} \frac{\ddot{r}^j(t)}{r^j(t)} = -R^j_{\beta j\delta} v^\beta v^\delta = -R^j_{tjt}.$$

No sum over j is implied in the above expression.

Since the volume of our ball is proportional to the product of the radii, $\ddot{V}/V \to \sum_j \ddot{r}^j/r^j$ as $t \to 0$, so

$$\lim_{V \to 0} \frac{\ddot{V}}{V}\Big|_{t=0} = -R^\alpha_{t\alpha t},$$

where now a sum over α is implied. The sum over α can range over all four coordinates, not just the three spatial ones, since the symmetries of the Riemann tensor demand that $R^t_{ttt} = 0$.

The right-hand side is minus the time-time component of the 'Ricci tensor'

$$R_{\beta\delta} = R^\alpha_{\beta\alpha\delta}.$$

That is,

$$\lim_{V \to 0} \frac{\ddot{V}}{V}\Big|_{t=0} = -R_{tt} \tag{7}$$

in local inertial coordinates where the ball starts out at rest.

In short, the Ricci tensor says how our ball of freely falling test particles starts changing in volume. The Ricci tensor only captures some of the information in the Riemann curvature tensor. The rest is captured by something called the 'Weyl tensor', which says how any such ball starts changing in shape. The Weyl tensor describes tidal forces, gravitational waves and the like.

Now, Einstein's equation in its usual form says

$$G_{\alpha\beta} = T_{\alpha\beta}.$$

Here the right side is the stress-energy tensor, while the left side, the 'Einstein tensor', is just an abbreviation for a quantity constructed from the Ricci tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{\gamma}_{\gamma}.$$

Thus Einstein's equation really says

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{\gamma}_{\gamma} = T_{\alpha\beta}. \tag{8}$$

This implies

$$R^{\alpha}_{\alpha} - \frac{1}{2} g^{\alpha}_{\alpha} R^{\gamma}_{\gamma} = T^{\alpha}_{\alpha},$$

but $g^{\alpha}_{\alpha} = 4$, so

$$-R^{\alpha}_{\alpha} = T^{\alpha}_{\alpha}.$$

Plugging this in equation (8), we get

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^{\gamma}_{\gamma}. \tag{9}$$

This is equivalent version of Einstein's equation, but with the roles of R and T switched! The good thing about this version is that it gives a formula for the Ricci tensor, which has a simple geometrical meaning.

Equation (9) will be true if any one component holds in all local inertial coordinate systems. This is a bit like the observation that all of Maxwell's equations are contained in Gauss's law and $\nabla \cdot B = 0$. Of course, this is only true if we know how the fields transform under change of coordinates. Here we assume that the transformation laws are known. Given this, Einstein's equation is equivalent to the fact that

$$R_{tt} = T_{tt} - \frac{1}{2} g_{tt} T^{\gamma}_{\gamma} \tag{10}$$

in every local inertial coordinate system about every point. In such coordinates we have

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

so $g_{tt} = -1$ and

$$T^{\gamma}_{\gamma} = -T_{tt} + T_{xx} + T_{yy} + T_{zz}.$$

Equation (10) thus says that

$$R_{tt} = \frac{1}{2}(T_{tt} + T_{xx} + T_{yy} + T_{zz}).$$

By equation (7), this is equivalent to

$$\lim_{V \to 0} \frac{\ddot{V}}{V}\Big|_{t=0} = -\frac{1}{2}(T_{tt} + T_{xx} + T_{yy} + T_{zz}).$$

As promised, this is the simple, tensor-calculus-free formulation of Einstein's equation.

6 Bibliography

Websites

There is a lot of material on general relativity available online, including many books, such as this:

Lecture Notes on General Relativity, S. M. Carroll,
http://arxiv.org/abs/gr-qc/9712019

The free online journal *Living Reviews in Relativity* is an excellent way to learn more about many aspects of relativity. One can access it at:

Living Reviews in Relativity, http://www.livingreviews.org

Part of learning relativity is working one's way through certain classic confusions. The most common are dealt with in the "Relativity and Cosmology" section of this site:

Frequently Asked Questions in Physics, edited by D. Koks
http://math.ucr.edu/home/baez/physics/

Nontechnical Books

Before diving into the details of general relativity, it is good to get oriented by reading some less technical books. Here are four excellent ones written by leading experts on the subject:

General Relativity from A to B, R. Geroch (University of Chicago Press, Chicago, 1981).

Black Holes and Time Warps: Einstein's Outrageous Legacy, K. S. Thorne (W. W. Norton, New York, 1995).

Gravity from the Ground Up: An Introductory Guide to Gravity and General Relativity, B. F. Schutz (Cambridge University Press, Cambridge 2003).

Space, Time, and Gravity: the Theory of the Big Bang and Black Holes, R. M. Wald (University of Chicago Press, Chicago, 1992).

Special Relativity

Before delving into general relativity in a more technical way, one must get up to speed on special relativity. Here are two excellent texts for this:

Introduction to Special Relativity, W. Rindler (Oxford University Press, Oxford, 1991).

Spacetime Physics: Introduction to Special Relativity, E. F. Taylor and J. A. Wheeler (W. H. Freeman, New York, 1992).

Introductory Texts

When one is ready to tackle the details of general relativity, it is probably good to start with one of these textbooks:

Introducing Einstein's Relativity, R. A. D'Inverno (Oxford University Press, Oxford, 1992).

Gravity: An Introduction to Einstein's General Relativity, J. B. Hartle (Addison-Wesley, New York, 2002).

Introduction to General Relativity, L. Hughston and K. P. Tod (Cambridge University Press, Cambridge, 1991).

A First Course in General Relativity, B. F. Schutz (Cambridge University Press, Cambridge, 1985).

General Relativity: An Introduction to the Theory of the Gravitational Field, H. Stephani (Cambridge University Press, Cambridge, 1990).

More Comprehensive Texts

To become a expert on general relativity, one really must tackle these classic texts:

Gravitation, C. W. Misner, K. S. Thorne, and J. A. Wheeler (W. H. Freeman, New York, 1973).

General Relativity, R. M. Wald (University of Chicago Press, Chicago, 1984).

Along with these textbooks, you'll want to do lots of problems! This book is a useful supplement:

Problem Book in Relativity and Gravitation, A. Lightman and R. H. Price (Princeton University Press, Princeton, 1975).

Experimental Tests

The experimental support for general relativity up to the early 1990s is summarized in:

Theory and Experiment in Gravitational Physics, Revised Edition, C. M. Will (Cambridge University Press, Cambridge, 1993).

A more up-to-date treatment of the subject can be found in:

"The Confrontation between General Relativity and Experiment," C. M. Will, *Living Reviews in Relativity* 4, 4 (2001). Available online at:
http://www.livingreviews.org/lrr-2001-4

Differential Geometry

The serious student of general relativity will experience a constant need to learn more tensor calculus — or in modern terminology, 'differential geometry'. Some of this can be found in the texts listed above, but it is also good to read mathematics texts. Here are a few:

Gauge Fields, Knots and Gravity, J. C. Baez and J. P. Muniain (World Scientific, Singapore, 1994).

An Introduction to Differentiable Manifolds and Riemannian Geometry, W. M. Boothby (Academic Press, New York, 1986).

Semi-Riemannian Geometry with Applications to Relativity, B. O'Neill (Academic Press, New York, 1983).

Specific Topics

The references above are about general relativity as a whole. Here are some suggested starting points for some of the particular topics touched on in this article.

The meaning of Einstein's equation. Feynman gives a quite different approach to this in:

The Feynman Lectures on Gravitation, R. P. Feynman et al. (Westview Press, Boulder, Colorado, 2002).

His approach focuses on the curvature of space rather than the curvature of spacetime.

The Raychaudhuri equation. This equation, which is closely related to our formulation of Einstein's equation, is treated in some standard textbooks, including the one by Wald mentioned above. A detailed discussion can be found in

Gravitation and Inertia, I. Ciufolini and J. A. Wheeler (Princeton University Press, Princeton, 1995).

Gravitational waves. Here are two nontechnical descriptions of the binary pulsar work for which Hulse and Taylor won the Nobel Prize:

"The Binary Pulsar: Gravity Waves Exist," C. M. Will, *Mercury*, Nov-Dec 1987, p. 162.

"Gravitational Waves from an Orbiting Pulsar," J. M. Weisberg, J. H. Taylor, and L. A. Fowler, *Scientific American*, Oct 1981, p. 74.

Here is a review article on the ongoing efforts to directly detect gravitational waves:

"Detection of Gravitational Waves," J. Lu, D. G. Blair, and C. Zhao, *Reports on Progress in Physics*, 63, 1317-1427 (2000).

Some present and future experiments to detect gravitational radiation are described here:

LIGO Laboratory Home Page, http://www.ligo.caltech.edu/

The Virgo Project, http://www.virgo.infn.it/

Laser Interferometer Space Antenna, http://lisa.jpl.nasa.gov/

Black holes. Astrophysical evidence that black holes exist is summarized in:

"Evidence for Black Holes," M. C. Begelman, *Science*, 300, 1898-1903 (2003).

A less technical discussion of the particular case of the supermassive black hole at the center of our Milky Way Galaxy can be found here:

The Black Hole at the Center of Our Galaxy, F. Melia (Princeton University Press, Princeton, 2003).

Cosmology. There are lots of good popular books on cosmology. Since the subject is changing rapidly, pick one that is up to date. At the time of this writing, we recommend:

The Extravagant Universe: Exploding Stars, Dark Energy, and the Accelerating Cosmos, R. P. Kirshner (Princeton University Press, Princeton, 2002).

A good online source of cosmological information is:

Ned Wright's Cosmology Tutorial,
http://www.astro.ucla.edu/~wright/cosmolog.htm

The following cosmology textbooks are arranged in increasing order of technical difficulty:

Cosmology: The Science of the Universe, 2nd ed., E. Harrison (Cambridge University Press, Cambridge, 2000).

Cosmology: a First Course, M. Lachièze-Rey (Cambridge University Press, Cambridge, 1995).

Principles of Physical Cosmology, P. J. E. Peebles (Princeton University Press, Princeton, 1993).

The Early Universe, E. W. Kolb and M. S. Turner (Addison–Wesley, New York, 1990).

The Large-Scale Structure of Spacetime, S. W. Hawking and G. F. R. Ellis (Cambridge University Press, Cambridge, 1975).

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